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Single-centered black hole microstate degeneracies from instantons in supergravity

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ABSTRACT: We obtain holographic constraints on the microscopic degeneracies of black holes by computing the exact macroscopic quantum entropy using localization, including the effects of string worldsheet instantons in the supergravity effective action. For $\frac{1}{4}$ -BPS black holes in type II string theory on $K3 \times T^2$, the constraints can be explicitly checked against expressions for the microscopic BPS counting functions that are known in terms of certain mock modular forms. We find that the effect of including the infinite sum over instantons in the holomorphic prepotential of the supergravity leads to a sum over Bessel functions with successively sub-leading arguments as in the Rademacher expansion of Jacobi forms — but begins to disagree with such a structure near an order where the mock modular nature becomes relevant. This leads to a systematic method to recover the polar terms of the microscopic degeneracies from the degeneracy of instantons (the Gromov-Witten invariants). We check explicitly that our formula agrees with the known microscopic answer for the first seven values of the magnetic charge invariant.

KEYWORDS: Black Holes, Black Holes in String Theory, Supergravity Models, Supersymmetric gauge theory

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1 Introduction and summary

Accounting for the thermodynamic entropy of a black hole as the statistical entropy of a microscopic ensemble has been one of the important successes of string theory. This idea can be made very precise in theories with supersymmetry because we can decouple the near-horizon configuration of a BPS black hole as an independent quantum system with AdS_2 boundary conditions. In particular, we can define the exact quantum entropy of a black hole as a functional integral in the gravitational theory with boundary conditions set by the attractor mechanism [1]. The exact computation of such functional integrals has been made possible due to the powerful technique of localization [2] applied to supergravity [3–5]. Although some important hurdles remain to be crossed in this program, we can — with some explicitly-stated assumptions — begin to write formulas for the perturbatively exact quantum entropy of a supersymmetric black hole in string theories with 8 supercharges in four dimensions [6]. We can thus compare the microscopic and macroscopic entropy formulas at a very precise level, pushing forward the earlier ideas of [7–15].

The best understood situation is that of $\frac{1}{8}$ -BPS black holes in maximally supersymmetric ($\mathcal{N} = 8$) theories. Localization reduces the full perturbative path integral in these theories to a one-dimensional integral which is simply the integral representation of a modified I -Bessel function. Going further, one can also identify all non-perturbative saddle-points of the full path integral [16, 17] and compute the contributions of fluctuations around them [18]. The exact non-perturbative expression for the black hole entropy is thus given by an infinite sum over different saddle-points yielding a corresponding infinite sum over I -Bessel functions with successively suppressed arguments, which add up to precisely the integer degeneracies of the microscopic ensemble computed in [19].

This remarkable manner in which continuum gravity arranges integer black hole degeneracies relies on the equally remarkable successive approximation of an integer in terms of complex analytic functions — eventually arriving at a convergent analytic series. This formula is well-known in analytic number theory as the *Hardy-Ramanujan-Rademacher expansion*. It is a consequence of the modular symmetry of the corresponding microscopic ensemble of the black hole constituents. This modular symmetry of the black hole ensemble is, however, special to $\mathcal{N} = 8$ string theory. In theories with lower supersymmetry, there are gravitational configurations other than the black hole that contribute to the full entropy formula [15, 20] (unlike the case for $\mathcal{N} = 8$ string theories [21]), and isolating the microstates belonging to the black hole will, in general, destroy modularity.

We have learnt about many aspects of the modular behavior of the microscopic partition functions in the generic setting of $\mathcal{N} = 2$ theories based on the modular nature of the effective strings when black holes descend from wrapped strings, and from the spacetime duality symmetries of the underlying theory [15, 22–28]. However, the counting function of microstates of a *single* black hole is still not understood in general, and in particular, it is not clear to what extent the modular symmetry of the original counting function has any remnant in the single-center black holes. In this paper we begin to address this problem from the point of view of the bulk gravitational theory.

The main point that we make here is that localization allows us to compute the perturbatively exact macroscopic formula for the black hole entropy. This formula is a very good analytic approximation to the microscopic degeneracies of the single-center black hole, and thus constrains the modular nature of their generating function. Under explicit assumptions about the prepotential and the functional integral measure in the language of effective supergravity, the exact macroscopic entropy has a structure similar to the Rademacher expansion of modular forms. As was already derived in [12, 15], following the OSV formula [9], the leading approximation to the degeneracy is given by a Bessel function with argument equal to a quarter of the area of the black hole, in the two-derivative approximation to the Wilsonian effective action of supergravity. Here we go beyond the leading order and show that including the infinite series of instanton effects in the holomorphic prepotential leads to a finite series of sub-dominant Bessel functions.

We illustrate this formula in the concrete setting of the $\mathcal{N} = 4$ string theory obtained as a type II compactification on $K3 \times T^2$. In this situation we have a complete knowledge of the non-perturbative prepotential in the supergravity theory, as well as that of the microscopic BPS counting function for $\frac{1}{4}$ -BPS states (see [29]). Further, it is known [21] that the only configurations, apart from dyonic $\frac{1}{4}$ -BPS black holes, that contribute to the relevant supersymmetric index are two-centered black holes which are each $\frac{1}{2}$ -BPS. Subtracting this two-centered contribution leads, as expected, to a breaking of modular symmetry for the single-centered black hole degeneracies of interest. It was shown in [30] that this breaking of modular symmetry happens in a very special manner and the single-centered black hole degeneracies are coefficients of mock modular forms [31, 32]. As a consequence, analytic number-theoretic expressions for the degeneracies can be resurrected — at the expense of some modifications to the formula due to the mock nature of the partition functions [33].

We find that the macroscopic answer in the $K3 \times T^2$ theory has the following structure. The prepotential of the theory is exact at one-loop order. The one-loop contribution to the prepotential depends only on a special modulus in the theory $S = -iX^1/X^0$, and it can be expanded as an infinite series in powers of the type e^{-nS} , where n is identified as the instanton number. The zero-instanton sector gives rise to the leading I -Bessel-function in the Rademacher expansion of the microscopic theory. In addition, the contribution from each of the infinite instanton sectors has the right structure to be identified with an I -Bessel function — seemingly leading to a badly divergent contribution to the answer. However, the choice of integration contour ensures that one gets sub-leading I -Bessel functions only until a certain value of the instanton number, beyond which one obtains exponentially suppressed terms.

The supergravity partition function can thus be expressed as a sum of Bessel functions with successively sub-leading arguments, with exactly the same arguments of the Bessels as those that appear in the Rademacher expansion of a Jacobi form. Quite remarkably, we find that the coefficients of the Bessel functions also agree exactly for the first many Bessel functions — and begin to deviate from the Rademacher expansion of a true Jacobi form exactly when we expect them to do so based on the above analysis due to the mock modular nature! This shows that the supergravity answer is sensitive to the polar coefficients of the microscopic function *including* the coefficients of the *mock modular* part. This looks to be the beginning of the answer to the question “How does the continuum supergravity know about the mock modular nature of the black hole partition function?”

Going further, we find a compact formula (4.25) for the polar terms of the microscopic black hole degeneracies in terms of the degeneracies of the worldsheet instantons. As we know quite well by now, the polar terms of a modular form completely control the full function, so our formula implies that we can predict all the black hole degeneracies purely from the knowledge of the worldsheet instanton degeneracies in the effective action of gravity. This idea — that black hole degeneracies can be derived in terms of the instanton degeneracies (i.e. the Gromov-Witten invariants) — was of course one of the main themes of the OSV formula and its refinements [15]. The fact that both the macroscopic and microscopic answers are known and are non-trivial in the $\mathcal{N} = 4$ situation gives us a nice set up to formulate and check these interesting ideas *exactly*. We stress that our formula agrees with the microscopic degeneracies of the *single-centered* black hole, i.e. only after we have subtracted the two-centered degeneracies from the full microscopic partition function.

We also point out a potential interest from a mathematical point of view — namely that our results look like the beginning of a consistent large-charge expansion for the coefficients of meromorphic Siegel modular forms which, in contrast with the Rademacher expansion for (mock) modular and Jacobi forms, is not really understood in the mathematics literature to the best of our knowledge. In order to complete this analysis, we need to classify and consider the effect of all gravitational saddle-points with AdS_2 boundary conditions (as was done in [18] for the $\mathcal{N} = 8$ theory). We leave this interesting problem for the future.

Our results thus bring us a step closer in the comparison of the microscopic and macroscopic exact degeneracy formulas for $\mathcal{N} = 4$ string theories. It would be interesting to extend this to more general $\mathcal{N} = 2$ situations. In particular, good progress has been made

in computing the exact prepotential in F-theory compactifications on elliptically fibered Calabi-Yau three-folds [34] which indeed has a special modulus which acts as a modular parameter. The problem of disentangling the exact single-centered black hole degeneracy was studied in this context in [35], but a full answer was not found. It would be very interesting if the macroscopic constraints we bring in threw new light on this problem.

This paper is organized as follows. In section 2 we present the microscopic counting formula for the quantum degeneracies of single-centered BPS black holes in $\mathcal{N} = 8$ and $\mathcal{N} = 4$ string theories, in terms of Jacobi forms, and mock Jacobi forms, respectively. In section 3 we present the macroscopic quantum entropy formula in supergravity computed using localization methods and show that the leading Bessel function is correctly reproduced from the tree-level Wilsonian effective action in the functional integral. In section 4 we include the instanton effects into the functional integral, and show that there is a finite series of sub-leading Bessel functions which has the same structure as the Rademacher expansion of Jacobi forms. In section 5 we perform an exact comparison of this expansion with the microscopic degeneracy formula and show how the supergravity theory is sensitive to the mock nature. In appendix A we present some facts about mock Jacobi forms that we use in this paper. In appendix B we discuss some details of the contour prescription that we use in section 4.

Note added. While this paper was being prepared for publication, we received [36] on the arXiv. As we use some of the technical analysis presented in that paper, we present a brief review of the relevant parts in section 4 and a refinement of some of the details in appendix B. We note that the aims and the main results of the two papers concern different topics. In particular, we do not aim to derive the measure of the supergravity localization formula in the present paper. Our focus here, as sketched above, is to derive an explicit formula relating the instanton degeneracies in supergravity and the microscopic black hole degeneracies.

2 Single-center black hole degeneracy and (mock) Jacobi forms

In this section we introduce the microscopic degeneracy formulas for the supersymmetric black holes that we study in this paper. We then present some details of the automorphic symmetry properties of the corresponding generating functions, which leads to analytic formulas for the degeneracies of a single-centered black hole. We begin by discussing the well-understood example of $\frac{1}{8}$ -BPS black holes in $\mathcal{N} = 8$ string theory as an illustration of the ideas. In this case the degeneracies are Fourier coefficients of a classical Jacobi form. We then move to the $\frac{1}{4}$ -BPS black holes in $\mathcal{N} = 4$ string theory where there are subtleties due to wall-crossing and the black hole degeneracies are coefficients of mock Jacobi forms. Here we will review the statements relevant to this paper and refer the reader interested in more details of these functions to [30].

$\frac{1}{8}$ -BPS black holes in $\mathcal{N} = 8$ string theory. We begin by considering type II string theory compactified on T^6 . At low energies the effective description of the theory is given by $\mathcal{N} = 8$ supergravity in four dimensions. This theory has a macroscopic $\frac{1}{8}$ -BPS black

hole solution carrying electric and magnetic charges under the various gauge fields in the theory. The $\mathcal{N} = 8$ string theory has an $E_{7,7}(\mathbb{Z})$ duality group with a duality invariant Δ that is quartic in the charges, and the entropy of the black hole depends only on Δ . In order to compute the microscopic degeneracies, one goes to a particular duality frame in which there is an explicit description of the charges of the black hole as charges in the microscopic string theory. A simple description consists of at least four charges which can be represented as follows. Writing $T^6 = T^4 \times S^1 \times \tilde{S}^1$, one has a D1-brane and a D5-brane wrapped on S^1 with momentum n on S^1 and one unit of Kaluza-Klein monopole charge on \tilde{S}^1 . In addition, one can turn on a fifth charge ℓ which corresponds to the momentum around \tilde{S}^1 . The duality invariant in this configuration is $\Delta = 4n - \ell^2$.

Using this description one can compute the BPS partition function which is the generating function of the microscopic index of $\frac{1}{8}$ -BPS states in the theory

$$Z^{\text{BPS}}(\tau, z) = \sum_{n, \ell \in \mathbb{Z}} c(n, \ell) q^n \zeta^\ell, \quad (2.1)$$

which has a simple explicit form in terms of theta and eta functions [19]:

$$Z^{\text{BPS}}(\tau, z) = \varphi_{-2,1}(\tau, z) := \frac{\vartheta_1(\tau, z)^2}{\eta(\tau)^6}. \quad (2.2)$$

The black hole degeneracies are related to the index of $\frac{1}{8}$ -BPS states in the theory [37, 38] as:

$$d(n, \ell) = (-1)^\ell c(n, \ell). \quad (2.3)$$

Jacobi forms. The function $\varphi = \varphi_{-2,1}$ is an example of a Jacobi form of weight $k = -2$ and index $m = 1$ whose defining property is the following two transformations. It is “modular in τ ”, i.e. it transforms under the modular group as

$$\varphi\left(\frac{a\tau + b}{c\tau + d}, \frac{z}{c\tau + d}\right) = (c\tau + d)^k e^{\frac{2\pi i m c z^2}{c\tau + d}} \varphi(\tau, z) \quad \forall \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}(2; \mathbb{Z}), \quad (2.4)$$

and “elliptic in z ”, i.e. it transforms under the translations of z by $\mathbb{Z}\tau + \mathbb{Z}$ as

$$\varphi(\tau, z + \lambda\tau + \mu) = e^{-2\pi i m (\lambda^2\tau + 2\lambda z)} \varphi(\tau, z) \quad \forall \quad \lambda, \mu \in \mathbb{Z}. \quad (2.5)$$

These symmetry properties are very powerful and they give us great control over its Fourier coefficients

$$\varphi(\tau, z) = \sum_{n, \ell \in \mathbb{Z}} c(n, \ell) q^n \zeta^\ell. \quad (2.6)$$

As a simple example, the elliptic transformation property (2.5) implies that the Fourier coefficients of a Jacobi form of index m obey the property

$$c(n, \ell) = C_\ell(4nm - \ell^2), \quad \text{where } C_\ell(\Delta) \text{ depends only on } \ell \bmod 2m. \quad (2.7)$$

The coefficients $C_\mu(\Delta)$ for fixed μ are the Fourier coefficients of modular forms. Indeed the $\mathcal{N} = 8$ black hole degeneracies $d(\Delta)$ can be written in terms of the coefficients $C_\ell(\Delta)$ of the Jacobi form (2.2) as:

$$d(\Delta) = (-1)^{\Delta+1} C_\mu(\Delta), \quad \text{with } \mu = \Delta \bmod 2, \quad (2.8)$$

which is a manifestation of the physical U -duality symmetry.

The precise mathematical definition of Jacobi forms [39] includes some technical conditions on the growth of the Fourier coefficients, in addition to the transformation formulas (2.4), (2.5). There are two types of Jacobi forms that will be relevant to us in this paper. The first is a *weakly holomorphic* Jacobi form, which means that the Fourier expansion in (2.6) obeys $n \geq -n_0$ for a fixed positive n_0 . This implies that there are only a finite number of terms with Fourier coefficients for negative values of Δ , and these are called the *polar* terms in the Fourier expansion of the Jacobi form. The second type is that of *weak Jacobi forms* which means that $n_0 = 0$ above. We refer the reader to [39] for a detailed theory of these functions.

The modular transformation property (2.4) is so powerful that one has an analytic formula for all the coefficients of a Jacobi form in terms of its polar coefficients. This formula, called the Hardy-Ramanujan-Rademacher formula, takes the form of an infinite convergent sum of Bessel functions, and is established by the so-called circle method in analytic number theory (see [23] for a nice exposition). The formula for the coefficients $C_\ell(\Delta)$ of a Jacobi form of weight $w + 1/2$ and index m , with $\Delta = 4mn - \ell^2$, has the following form:

$$C_\ell(\Delta) = (2\pi)^{2-w} \sum_{c=1}^{\infty} c^{w-2} \sum_{\tilde{\ell} \in \mathbb{Z}/2m\mathbb{Z}} \sum_{\tilde{\Delta} < 0} C_{\tilde{\ell}}(\tilde{\Delta}) K\ell(\Delta, \ell, \tilde{\Delta}, \tilde{\ell}; c) \left| \frac{\tilde{\Delta}}{4m} \right|^{1-w} \tilde{I}_{1-w} \left(\frac{\pi}{mc} \sqrt{|\tilde{\Delta}| \Delta} \right), \quad (2.9)$$

where

$$\tilde{I}_\rho(z) = \frac{1}{2\pi i} \int_{\epsilon-i\infty}^{\epsilon+i\infty} \frac{d\sigma}{\sigma^{\rho+1}} \exp\left(\sigma + \frac{z^2}{4\sigma}\right), \quad (2.10)$$

is called the modified Bessel function of index ρ , and is related to the standard Bessel function of the first kind $I_\rho(z)$ by

$$\tilde{I}_\rho(z) = \left(\frac{z}{2}\right)^{-\rho} I_\rho(z). \quad (2.11)$$

The coefficients $K\ell(\Delta, \ell, \tilde{\Delta}, \tilde{\ell}; c)$ are the so-called generalized Kloosterman sums [23], and they are essentially sums of phases, thus carrying practically no entropy. For $c = 1$, they are given by:

$$K\ell(\Delta, \ell, \tilde{\Delta}, \tilde{\ell}; c = 1) = S_{\tilde{\ell}\ell}^{-1} = \sqrt{\frac{2}{m}} e^{i\pi(m-\ell')\frac{\ell}{m}}. \quad (2.12)$$

The remarkable thing about Formula (2.9) is that the coefficients $C_\ell(\Delta)$ for $\Delta > 0$ are completely determined by the coefficients $C_{\tilde{\ell}}(\tilde{\Delta})$ associated to the so-called polar terms $q^{\tilde{\Delta}}$ with $\tilde{\Delta} < 0$, which are finite in number. The asymptotic formula of the Bessel function $I(z) \sim e^z$ for large z shows that the terms with $c > 1$ are exponentially suppressed compared to the leading $c = 1$ terms.

The $\mathcal{N} = 8$ black hole partition function $\varphi_{-2,1}$ (2.2) is a weak Jacobi form. There is only one polar coefficient with $\tilde{\Delta} = -1$, and so the Rademacher formula simplifies to:

$$C(\Delta) = 2\pi \left(\frac{\pi}{2}\right)^{7/2} \sum_{c=1}^{\infty} c^{-9/2} K_c(\Delta) \tilde{I}_{7/2} \left(\frac{\pi\sqrt{\Delta}}{c} \right). \quad (2.13)$$

Here K_c is a particular combination of the Kloosterman sums with the property $K_1(\Delta) = 1$. As was shown in [5], the leading $c = 1$ Bessel function can be recovered as the exact functional integral of supergravity with $AdS_2 \times S^2$ boundary conditions. It can thus be interpreted as the all-order perturbation theory result for the quantum entropy, including all perturbative quantum corrections to the Bekenstein-Hawking formula. The exponentially sub-leading terms with $c > 1$, including the Kloosterman sums, can also be recovered as exact functional integrals over different orbifold sectors of $AdS_2 \times S^2$ [18].

$\frac{1}{4}$ -BPS black holes in $\mathcal{N} = 4$ string theory. Now we come to the $\mathcal{N} = 4$ theory. Consider type II string theory compactified on $K3 \times T^2$ or, equivalently, heterotic string theory on T^6 . At low energies the effective description of the theory is given by $\mathcal{N} = 4$ supergravity coupled to 28 $\mathcal{N} = 4$ gauge field multiplets specified by the compactification. The quarter-BPS black holes carry electric and magnetic charges (Q_e^i, Q_m^i) ($i = 1, \dots, 28$), under these gauge fields, where i is a vector index under the T-duality group $SO(6, 22)$, and (Q_e, Q_m) transform as a doublet under the S-duality group $SL(2, \mathbb{Z})$. The U-duality group of the theory is $SL(2, \mathbb{Z}) \times SO(6, 22)$.

One-fourth BPS dyonic states in the theory are completely labelled by the three continuous T-duality invariants:

$$(Q_e^2/2, Q_e \cdot Q_m, Q_m^2/2) \equiv (n, \ell, m), \quad (2.14)$$

and, in addition, some discrete charge invariants [40]. As for the $\mathcal{N} = 8$ example we write the compactification manifold as $K3 \times S^1 \times \tilde{S}^1$, and we can choose a duality frame in which the black hole consists of the D1-D5-P system wrapping $K3 \times S^1$ with Q_1 D1-branes, Q_5 D5-branes, n units of momentum on S^1 , one unit of KK-monopole charge and ℓ units of momentum on \tilde{S}^1 . The charge invariants are $(Q_e^2/2 = n, Q_e \cdot Q_m = \ell, Q_m^2/2 = Q_1 Q_5)$. The exact microscopic counting formula for the index of a generic one-fourth BPS state has been worked out completely [41–45]. For charges where the discrete invariants are trivial, the BPS indexed partition function is given by

$$Z^{\text{BPS}}(\tau, z, \sigma) = \frac{1}{\Phi_{10}(\tau, z, \sigma)}, \quad (2.15)$$

where we now have three chemical potentials that couple to the three T-duality invariants. The function Φ_{10} is the Igusa cusp form, which is the unique Siegel cusp form of weight 10. The microscopic degeneracy is given by the so-called DVV formula [41]:

$$d(n, \ell, m) = (-1)^{\ell+1} \int_{\mathcal{C}} d\tau dz d\sigma \frac{e^{-i\pi(\tau n + 2z\ell + \sigma m)}}{\Phi_{10}(\tau, z, \sigma)}, \quad (2.16)$$

with a contour \mathcal{C} that was spelled out in [46].

Mock Jacobi forms. There is an important new physical phenomenon that arises in the $\mathcal{N} = 4$ theory as compared to the $\mathcal{N} = 8$ theory. While the microscopic index that counts one-eighth-BPS states in the $\mathcal{N} = 8$ theory only gets contributions from single-centered black holes, the corresponding index that counts quarter-BPS states in the $\mathcal{N} = 4$

theory gets contributions from single-centered black holes as well as two-centered black hole configurations, depending on the value of the moduli at infinity [21]. This ambiguity is captured in the DVV formula by the choice of contour in (2.16), which depends on the moduli fields at infinity [46, 47]. Choosing the moduli to be at the attractor point yields the pure single-centered black hole degeneracies. Doing so, however, destroys the modular symmetry. From a physical point of view this breaking is related to the fact that we are throwing away a part of the spectrum of the theory. From a mathematical point of view it is because the partition function $1/\Phi_{10}$ is a *meromorphic* function with poles in the bulk of the Siegel upper half plane.

Without the powerful handle given by the modular symmetry, it looks at first sight like the program followed to interpret the microscopic degeneracies in supergravity will not work. In particular, we do not know how to write down an analytic expansion like (2.13) for the $\mathcal{N} = 8$ black hole case. This problem was solved in [30] as we now briefly summarize. (We give more details in appendix A.) We can perform one of the three Fourier expansions in (2.16) near $\sigma \rightarrow i\infty$ to obtain:

$$\frac{1}{\Phi_{10}(\tau, z, \sigma)} = \sum_{m \geq -1} \psi_m(\tau, z) e^{2\pi i m \sigma}. \quad (2.17)$$

The functions ψ_m are Jacobi forms of weight -10 and index m that are meromorphic (in z). These contain the degeneracies of states with magnetic charge m , including both single and two-centered black holes. The single-centered black hole degeneracies are found by subtracting the generating function of two-centered degeneracies (called ψ_m^P) from ψ_m . The difference, called the finite or Fourier part of ψ_m

$$\psi_m^F = \psi_m - \psi_m^P,$$

is holomorphic in z , and has an unambiguous Fourier expansion:

$$\psi_m^F(\tau, z) = \sum_{n, \ell} c_m^F(n, \ell) q^n \zeta^\ell. \quad (2.18)$$

It was shown in [30] that:

- (i) The microscopic indexed degeneracies $d(n, \ell, m)$ of the single-centered black holes (i.e. corresponding to the attractor contour) are precisely related to the Fourier coefficients of this function

$$d(n, \ell, m) = (-1)^{\ell+1} c_m^F(n, \ell), \quad (2.19)$$

- (ii) The function $\psi_m^F(\tau, z)$ is a *mock Jacobi form*.

The meaning of the word *mock* is that the transformation rule (2.4) is modified. The functions ψ_m^F themselves are not modular, but one can add a correction term called the *shadow* to get completed functions $\widehat{\psi}_m^F$ that are modular, i.e. they transform exactly with

the rule (2.4). The shadow is a non-holomorphic function¹ and leads to a holomorphic anomaly equation as in (A.9). This resurrection of modular symmetry means, in particular, that we can again use the circle method to get a formula for the Fourier coefficients. This formula differs from that of the analogous formula for true Jacobi forms (the Rademacher expansion) due to the effect of the shadow term (see [33, 55]). In order to make sharp estimates about how the asymptotic expansion of mock Jacobi forms differs from that of true Jacobi forms, we need to know the explicit expressions of the mock Jacobi forms in question. This is a fairly complicated question but it has been addressed and solved in ([30], Chapters 9, 10). We provide some relevant details in appendix A, and here we illustrate the main points with some examples.

In order to present the results, we need to introduce two Jacobi forms

$$A(\tau, z) = \varphi_{-2,1}(\tau, z) := \frac{\vartheta_1^2(\tau, z)}{\eta^6(\tau)}, \quad (2.20)$$

$$B(\tau, z) = \varphi_{0,1}(\tau, z) := 4 \left(\frac{\vartheta_2^2(\tau, z)}{\vartheta_2^2(\tau)} + \frac{\vartheta_3^2(\tau, z)}{\vartheta_3^2(\tau)} + \frac{\vartheta_4^2(\tau, z)}{\vartheta_4^2(\tau)} \right), \quad (2.21)$$

where $\vartheta_i, i = 1, \dots, 4$ are the four classical Jacobi theta functions. These two Jacobi forms generate the ring of all weak Jacobi forms of even weight over the ring of modular forms [39]. The word “weak” here refers to a growth condition on the functions, and it means in particular that for large values of $\Delta = 4mn - \ell^2$, the coefficients grow as (see appendix A)

$$c(n, \ell) \simeq \exp(\pi \sqrt{4mn - \ell^2}). \quad (2.22)$$

The functions ψ_m^F can be worked out explicitly (see [55]) for a given value of m . The first couple of cases are:

$$\psi_1^F = \frac{1}{\eta(\tau)^{24}} (3E_4 A - 648\mathcal{H}_1), \quad (2.23)$$

$$\psi_2^F = \frac{1}{3\eta(\tau)^{24}} (22E_4 AB - 10E_6 A^2 - 9600\mathcal{H}_2). \quad (2.24)$$

Here the functions $\mathcal{H}_1, \mathcal{H}_2$ are mock Jacobi forms whose coefficients are linear combinations of the so-called Hurwitz-Kronecker class numbers, whose Fourier coefficients have purely polynomial growth. This is representative of the general structure proved in [30]: the mock Jacobi forms ψ^F can always be written as a sum of two pieces: $\varphi_{2,m}^{\text{true}}(\tau, z)/\eta(\tau)^{24}$ and $\varphi_{2,m}^{\text{opt}}(\tau, z)/\eta(\tau)^{24}$. The function $\varphi_{2,m}^{\text{true}}(\tau, z)$ is a true weak Jacobi form (in particular, we can apply the usual Rademacher expansion (2.9) to it), and the second is a mock Jacobi form of a very special kind in that its Fourier coefficients grow extremely slowly. In the two examples above, this growth is purely polynomial — that is the case whenever m is a prime power. In general, the growth of the coefficients of $\varphi_{2,m}^{\text{opt}}(\tau, z)$ goes as

$$c^{\text{opt}}(n, \ell) \sim \exp\left(\frac{\pi}{m} \sqrt{4mn - \ell^2}\right). \quad (2.25)$$

¹See [48–54] for the physical origin of such non-holomorphic terms from the point of view of conformal field theory. Understanding the physical basis of the non-holomorphicity of the specific functions $\widehat{\psi}_m^F$ is an interesting open problem.

which can be contrasted with (2.22). What we need is to estimate the growth of the ratios like $\varphi_{2,m}^{\text{opt}}(\tau, z)/\eta(\tau)^{24}$ that enter our expressions. Such functions are called *mixed* mock Jacobi forms, and their Rademacher expansion already differs at leading order in the asymptotic expansion compared to a true Jacobi form of the same weight and index (see Comment 1 below Theorem (1.3) of [33]).

We are now ready to reap the benefits of this technical analysis. If we want to analyze the Rademacher expansion of the black hole degeneracies encoded in ψ_m^{F} , we can use the usual Rademacher expansion (2.9) of Jacobi forms *as long as* the growth of Bessel functions in (2.9) are larger than the growth of the mixed mock Jacobi forms $\varphi_{2,m}^{\text{opt}}(\tau, z)/\eta(\tau)^{24}$. From what we said above, it is clear that we always have the contribution of the (denoting polynomial prefactors by p_i for now)

$$\text{Leading Bessel:} \quad p_0 \tilde{I}_{23/2} \left(2\pi \sqrt{(m+4) \left(n - \frac{\ell^2}{4m} \right)} \right), \quad (2.26)$$

where $p_0 = (m+2) \frac{4\pi}{\sqrt{m}} \left(\frac{m+4}{n - \frac{\ell^2}{4m}} \right)^{23/4}$ as for a true Jacobi form for any m . This is then followed by the sub-leading Bessel functions in the $c=1$ series of (2.9):

$$\begin{aligned} \text{Sub-leading } c=1 \text{ series:} \quad & p_1 \tilde{I}_{23/2} \left(2\pi \sqrt{\left(\frac{(m-1)^2}{m} + 4 \right) \left(n - \frac{\ell^2}{4m} \right)} \right) + \\ & p_2 \tilde{I}_{23/2} \left(2\pi \sqrt{\left(\frac{(m-2)^2}{m} + 4 \right) \left(n - \frac{\ell^2}{4m} \right)} \right) + \dots \end{aligned} \quad (2.27)$$

But we should stop trusting this series when one of two things happen: firstly the $c=2$ term begins to contribute at the order

$$c=2 \text{ series:} \quad \tilde{I}_{23/2} \left(2\pi \sqrt{\frac{(m+4)}{4} \left(n - \frac{\ell^2}{4m} \right)} \right). \quad (2.28)$$

Secondly the mock modular terms begin to contribute according to the discussion above. We need to use a modified Rademacher expansion for the mixed mock Jacobi forms as in [33]. Working out the details of the latter is an interesting problem in analytic number theory which we leave for the future (and for the experts!).

We will use this analysis in section 5 to work out some details of when exactly the signature of the mock nature appears in the Rademacher expansion on a case-by-case basis for the first few values of m . We now change track and move on to a supergravity analysis of the single-center black hole partition function.

3 Localization in supergravity

We begin this section with a review of the exact computation of the quantum entropy of BPS black holes in four-dimensional $\mathcal{N}=2$ supergravity following [4, 5]. We consider the particular case of $\frac{1}{4}$ -BPS black holes in $\mathcal{N}=4$ string theory coming from the compactification of Type II string theory on $K3 \times T^2$. In the two-derivative limit of supergravity, we

show how the functional integral in the near-horizon AdS_2 reduces to a single Bessel function. We then set the stage for the inclusion of instantons in the holomorphic prepotential of the supergravity, which we will treat in the next section.

The theory under consideration is described by $\mathcal{N} = 2$ superconformal gravity in 4 dimensions with the Weyl multiplet coupled to $n_v + 1$ vector multiplets. The Weyl multiplet contains the vielbein, the gravitini, and auxiliary fields including an antisymmetric tensor $T_{\mu\nu}$ with self-dual and anti-self dual components denoted by $T_{\mu\nu}^\pm$. Each vector multiplet $I = 0, \dots, n_v$, contains a vector field A_μ^I , a complex scalar X^I , the gaugini, and three auxiliary fields Y_j^{Ii} , $i, j = 1, 2$, with $Y_1^{I1} = -Y_2^{I2}$. In this theory we consider a BPS black hole solution carrying electric and magnetic charges q_I, p^I . In order to compute the exact quantum entropy of this black hole, we use the supersymmetric localization technique applied to this problem [4, 5]. The near-horizon configuration of the classical black hole is found by the attractor mechanism to be the $AdS_2 \times S^2$ geometry with fixed electric and magnetic fields, and constant scalar fields:

$$\begin{aligned} ds^2 &= v(d\eta^2 + \sinh^2 \eta d\tau^2 + d\psi^2 + \sin^2 \psi d\phi^2), \\ F_{\eta\tau}^I &= ie_*^I \sinh \eta, \quad F_{\psi\phi}^I = p^I \sin \psi, \quad X^I = X_*^I. \end{aligned} \quad (3.1)$$

Here $X_*^I = \frac{1}{2}(e_*^I + ip^I)$ is the attractor value of the scalar field with e_*^I and p^I the electric field and the magnetic charge of the black hole, respectively. The electric field is related to the electric charge q_I of the black hole by a Legendre transform $q_I = \partial \mathcal{L} / \partial e_*^I$.

In order to compute the exact functional integral using localization, we first find the localization manifold \mathcal{M}_Q which is the locus of all solutions to the BPS equations. The results of [4, 56] show that only the scalar and auxiliary fields of the vector multiplets are excited on this localization manifold:

$$X^I = X_*^I + \frac{C^I}{\cosh \eta}, \quad \bar{X}^I = \bar{X}_*^I + \frac{C^I}{\cosh \eta}, \quad Y_1^{I1} = -Y_2^{I2} = \frac{2C^I}{\cosh^2 \eta}, \quad (3.2)$$

where the C^I are arbitrary real numbers, while all other fields stay fixed to their classical attractor values (3.1). It is convenient to label the localizing manifold by the variables:

$$\phi^I \equiv e_*^I + 2C^I. \quad (3.3)$$

The exact quantum entropy of the black hole, as derived in [4] is:

$$\widehat{W}(q, p) = \int_{\mathcal{M}_Q} \exp(\mathcal{S}_{\text{ren}}(\phi, q, p)) [d\mu(\phi)]. \quad (3.4)$$

Here $[d\mu(\phi)]$ is the induced measure from the supergravity, including the classical induced measure longitudinal to \mathcal{M}_Q , and a one-loop determinant coming from integration over non-BPS directions orthogonal to \mathcal{M}_Q . We will comment on this in the following. The function \mathcal{S}_{ren} is the renormalized action of the theory coming from evaluating the full supergravity action on the localizing solutions and following a regularization procedure [1, 4] to remove the divergences coming from the infinite volume of AdS_2 .

The full action of $\mathcal{N} = 2$ supergravity can be divided into two class — terms coming from chiral superspace integrals of the holomorphic prepotential, and those coming from full superspace integrals. It was shown in [57] that only the former type of terms contribute to the quantum entropy. These terms are encoded in one holomorphic function $F(X^I, \hat{A})$ describing the coupling of the vector multiplets to the background Weyl multiplet. Here, X^I is the lowest component of the vector multiplet and $\hat{A} \equiv (T_{\mu\nu}^-)^2$ is the lowest component of the chiral multiplet built as the square of the Weyl multiplet. The renormalized action on the localization manifold evaluated for the action governed by F takes the simple form:

$$\mathcal{S}_{ren}(\phi, q, p) = -\pi q_I \phi^I + 4\pi \text{Im} F\left(\frac{\phi^I + ip^I}{2}\right). \quad (3.5)$$

In this equation (and sometimes in the following), we have used the fact that the attractor equations set $\hat{A} = -64$. The prepotential function $F(X^I, \hat{A})$ is a homogeneous function of weight 2 under the scalings $X^I \rightarrow \lambda X^I$, $\hat{A} \rightarrow \lambda^2 \hat{A}$, and it can be expanded as:

$$F(X^I, \hat{A}) = \sum_{g=0}^{\infty} F^{(g)}(X^I) \hat{A}^g. \quad (3.6)$$

The function $F^{(0)}(X^I)$ controls the two-derivative interactions, and the coefficients $F^{(g)}$, $g \geq 1$, describe higher derivative couplings in the theory.

Now we consider the $K3 \times T^2$ compactification of the type II theory. Writing this theory as an $\mathcal{N} = 2$ supergravity yields a field content, in addition to the Weyl multiplet, of vector multiplets, hyper multiplets and gravitino multiplets. Following the ideas of [13] one can truncate this theory to an $\mathcal{N} = 2$ supergravity with a Weyl multiplet and $n_v = 23$ vector multiplets. In this case the perturbative prepotential has the form:

$$F^{\text{tree}}(X) = -\frac{X^1}{X^0} X^a C_{ab} X^b + \frac{X^1}{X^0}, \quad (3.7)$$

where C_{ab} is the intersection matrix of $K3$ on the middle homology. In the full theory, this is modified due to the effects of worldsheet instantons, as we shall consider in the following.

Within this set up one can solve the exact functional integral explicitly, as we now briefly recall. In this $\mathcal{N} = 2$ supergravity set up, the charge configuration corresponding to the microscopic configuration described below Equation (2.14) corresponds to a non-zero value of $(q_0, q_2, p^1, p^2, p^3)$ as discussed in [5]. It was argued in [4], based on the structure of the classical metric of the moduli space, that the induced measure on the localizing manifold in the large-charge limit is:

$$[d\mu(\phi)] = P_1 \frac{1}{p^1 \phi^0} \prod_I d\phi^I, \quad (3.8)$$

where the prefactor P_1 is a function only of the charges and independent of the coordinates ϕ^I . One generically expects the measure factor to change when we go beyond the tree-level approximation. We shall discuss this in the next section.

Under these assumptions, the quantum entropy (3.4) takes the form

$$\widehat{W}^{\text{tree}}(p, q) = P_1 \int \frac{d\phi^0 d\phi^1}{\phi^0 p^1} \exp(-\pi \phi^0 q_0) \int \prod_{a=2}^{n_v} d\phi^a \exp\left(-\pi \phi^2 q_2 + 4\pi \text{Im} F^{\text{tree}}\left(\frac{\phi^I + ip^I}{2}\right)\right). \quad (3.9)$$

From (3.7), we see that the last $(n_v - 1)$ integrals are Gaussian integrals which yield:

$$\begin{aligned} & \int \prod_{a=2}^{n_v} d\phi^a \exp\left(-\pi \phi^2 q_2 + 4\pi \text{Im} F^{\text{tree}}\left(\frac{\phi^I + ip^I}{2}\right)\right) \\ &= \left(\frac{\phi^0}{p^1}\right)^{(n_v-1)/2} \exp\left(\pi \frac{\phi^1}{p^1} p^1 q_2\right) \exp\left(\pi \frac{\phi^1}{\phi^0} \left(\frac{\phi^1}{p^1} + \frac{p^1}{\phi^1}\right) p^a C_{ab} p^b + 4\pi \frac{p^1}{\phi^0}\right). \end{aligned} \quad (3.10)$$

The change of variables $\tau_1 = \phi^1/\phi^0$, $\tau_2 = p^1/\phi^0$ yields

$$\widehat{W}^{\text{tree}}(p, q) = P_1 \int \frac{d\tau_1 d\tau_2}{\tau_2^{(n_v+3)/2}} \exp\left(\frac{\pi}{\tau_2} \left(-p^1 q_0 + p^1 q_2 \tau_1 + p^a C_{ab} p^b \tau_1^2 + (p^a C_{ab} p^b + 4) \tau_2^2\right)\right). \quad (3.11)$$

Upon identifying the four-dimensional electric and magnetic charge invariants as

$$Q_e^2/2 \equiv -q_0 p^1, \quad Q_m^2/2 \equiv p^a C_{ab} p^b, \quad Q_e \cdot Q_m \equiv -q_2 p^1, \quad (3.12)$$

and with the identification $(Q_e^2/2, Q_e \cdot Q_m, Q_m^2/2) = (n, \ell, m)$ as in Equation (2.14), this takes the form,

$$\widehat{W}^{\text{tree}}(n, \ell, m) = P_1 \int \frac{d^2 \tau}{\tau_2^{(n_v+3)/2}} \exp\left(\frac{\pi}{\tau_2} (n - \ell \tau_1 + m \tau_1^2 + (m+4) \tau_2^2)\right). \quad (3.13)$$

The τ_1 integral is Gaussian and can be evaluated in a straightforward manner. The remaining integral over τ_2 can be evaluated using the contour integral representation of the Bessel function (2.10),

$$\widehat{W}^{\text{tree}}(n, \ell, m) = P_1 \frac{2\pi}{\sqrt{m}} \left(\frac{m+4}{n - \frac{\ell^2}{4m}}\right)^{23/4} I_{23/2} \left(2\pi \sqrt{(m+4) \left(n - \frac{\ell^2}{4m}\right)}\right). \quad (3.14)$$

It has been argued recently in [36] that the prefactor $P_1 = 2m + 4$. The function (3.14) then agrees precisely with the leading Bessel function in the Rademacher expansion of the microscopic theory (2.9) with the right weight, argument, and prefactor.

Now we move to the instanton contributions. We note that we kept only the perturbative prepotential to first sub-leading order while in general we have instanton sums that generate an infinite series of corrections to the prepotential (3.7). In general the instantons contribute to all the couplings $F^{(g)}$. In the type II theory on $K3 \times T^2$ the holomorphic prepotential is one-loop exact:

$$F(X) = -\frac{X^1 X^a C_{ab} X^b}{X^0} + \frac{1}{2\pi i} \mathcal{F}_{K3 \times T^2}^{(1)}(X^1/X^0). \quad (3.15)$$

The one-loop contribution to the prepotential is:

$$\mathcal{F}_{K3 \times T^2}^{(1)}(X^1/X^0) = \log\left(\eta^{24}(X^1/X^0)\right), \quad (3.16)$$

and has the expansion

$$\mathcal{F}_{K3 \times T^2}^{(1)}(X^1/X^0) = 2\pi i \frac{X^1}{X^0} + \tilde{\mathcal{F}}^{\text{inst}}(X^1/X^0). \quad (3.17)$$

Here the function $\tilde{\mathcal{F}}^{\text{inst}}$ encodes the contributions of worldsheet instantons in the type II theory to the prepotential:

$$\tilde{\mathcal{F}}^{\text{inst}}(\tau) = -\log \prod_{n=1}^{\infty} (1 - e^{2\pi i n \tau})^{-24}. \quad (3.18)$$

A natural question is how to properly take these corrections coming from the instantons into account. The instantons can affect the exact answer (3.4) in two ways — by the explicit change of the prepotential in the exponent of (3.5), and by an implicit effect on the measure of the integral (which was also computed above in the zero-instanton sector). This is what we turn to in the following section.

4 Including instantons in the functional integral

In this section, we work out the corrections to (3.14) due to instantons. We write:

$$\widehat{W}(n, \ell, m) = \int_{\gamma} \frac{d^2 \tau}{\tau_2^{(n+3)/2}} e^{\frac{\pi}{\tau_2} (n - \ell \tau_1 + m \tau_1^2 + m \tau_2^2)} M(\tau, \bar{\tau}) e^{-\mathcal{F}^{(1)}(\tau) - \mathcal{F}^{(1)}(-\bar{\tau})}. \quad (4.1)$$

Here we have taken into account the explicit effect on the prepotential function:

$$F(X) = -\frac{X^1 X^a C_{ab} X^b}{X^0} + \frac{1}{2i\pi} \mathcal{F}^{(1)}\left(\frac{X^1}{X^0}\right), \quad (4.2)$$

with $\mathcal{F}^{(1)}$ given in (3.16) being the one-loop effect (which is exact in this case), which contains contributions from an infinite set of worldsheet instantons wrapping the torus. Naively the inclusion of all these instantons leads to an infinite series of I -Bessel functions. In this section we show that with an appropriate choice of contour γ in (4.1) most of these are in fact exponentially suppressed, leading to a *finite* number of Bessel functions that contribute to the quantum entropy. This finite sum has precisely the same structure as the leading $c = 1$ term in the expansion (2.9) for Jacobi forms.

We preface the calculation in this section with some remarks on the measure in Equation (4.1). We have parametrized the effect of instanton corrections on the measure of the functional integral by the function $M(\tau, \bar{\tau})$. In section 3 we did not take into account the full quantum effects on the measure in the localization computation. Indeed one needs to compute the one-loop determinant of the off-shell fluctuations of the non-BPS modes, and compute the induced measure from the supergravity field space. The former was computed explicitly in [6, 58] for matter multiplets in $\mathcal{N} = 2$ supergravity. Further it was argued that the symmetries fix the form of the graviton multiplet determinant up to a constant that can be matched to an on-shell computation [59]. The latter has been addressed in various papers [36, 60, 61] although we think it is fair to say that a full satisfactory

first-principles derivation of this measure has not been reached yet. We do not attempt to solve this problem in the current paper. Instead we will use the fact that one knows the exact measure factor based on a contour integral and a subsequent expansion of the DVV formula [13, 43, 62], as we shall now present. Note that this is a different expansion compared to [30] that is used to compute the exact single-centered degeneracies. In particular, the expansion of [30] explicitly subtracts the two-centered black hole contributions from the Siegel modular form that is the full $\frac{1}{4}$ -BPS partition function, and then for each magnetic charge invariant, throws up a mock Jacobi form (whose coefficients can be then again expanded in a Rademacher-type series). The formulas in this section, as we shall see below, follow from keeping only the residue at the leading divisor of the Siegel modular form. These two expansions are not a priori related, and so the results of this and the next section are non-trivial. They seem to point to a Rademacher-type expansion of the Siegel modular form, which was anticipated in [17].

We begin with the DVV formula (2.16) which is a three-dimensional contour integral:²

$$d(n, \ell, m) = (-1)^{\ell+1} \int_{\mathcal{C}} d\sigma dv d\rho \frac{e^{-i\pi(\sigma n + 2v\ell + \rho m)}}{\Phi_{10}(\rho, v, \sigma)}. \quad (4.3)$$

We can perform an exact contour integral in the v -variable which reduces to picking up residues at the divisors of $1/\Phi_{10}$ in the Siegel upper-half plane, leaving a two-dimensional integral over σ, ρ which are reexpressed as $\sigma = \tau_1 + i\tau_2$, $\rho = -\tau_1 + i\tau_2$. The result is [43]:

$$d(n, \ell, m) \simeq \int_{\gamma} \frac{d\tau_1 d\tau_2}{\tau_2^2} e^{-F(\tau_1, \tau_2)}, \quad (4.4)$$

where \simeq implies equality up to exponentially suppressed contributions coming from additional poles, which we shall suppress from now on. The function $F(\tau_1, \tau_2)$ is given by:

$$F(\tau_1, \tau_2) = -\frac{\pi}{\tau_2} (n - \ell\tau_1 + m(\tau_1^2 + \tau_2^2)) + \ln \eta^{24}(\tau_1 + i\tau_2) + \ln \eta^{24}(-\tau_1 + i\tau_2) + 12 \ln(2\tau_2) \\ - \ln \left[\frac{1}{4\pi} \left\{ 26 + \frac{2\pi}{\tau_2} (n - \ell\tau_1 + m(\tau_1^2 + \tau_2^2)) \right\} \right], \quad (4.5)$$

and the contour of integration γ is required to pass through the saddle-point of $F(\tau_1, \tau_2)$. We rewrite this formula by adding the following total derivative³ to the integrand of (4.4),

$$\frac{d}{d\tau_2} \left(\frac{1}{\tau_2^{13}} e^{\frac{\pi}{\tau_2} (n - \ell\tau_1 + m\tau_1^2 + m\tau_2^2) - \ln \eta^{24}(\tau_1 + i\tau_2) - \ln \eta^{24}(-\tau_1 + i\tau_2)} \right), \quad (4.6)$$

which yields (with $\tau = \tau_1 + i\tau_2$):

$$d(n, \ell, m) = \frac{1}{2^{12}} \int_{\gamma} \frac{d^2\tau}{\tau_2^{13}} (m + E_2(\tau) + E_2(-\bar{\tau})) (\eta^{24}(\tau) \eta^{24}(-\bar{\tau}))^{-1} e^{\frac{\pi}{\tau_2} (n - \ell\tau_1 + m\tau_1^2 + m\tau_2^2)}, \quad (4.7)$$

where E_2 is the Eisenstein series of weight 2 related to the Dedekind eta function as:

$$E_2(\tau) = \frac{1}{2\pi i} \frac{d}{d\tau} \log \eta^{24}(\tau). \quad (4.8)$$

²For the next few lines we will use the variables (σ, v, ρ) instead of (τ, z, σ) to avoid confusion.

³This trick was independently noted and used in [36].

Comparing this to our parametrization (4.1), we obtain:

$$M(\tau, \bar{\tau}) = \frac{1}{2^{12}}(m + E_2(\tau) + E_2(-\bar{\tau})). \quad (4.9)$$

We note that M can be written, as anticipated in [12, 13], in terms of the generalized Kähler potential defined as

$$e^{-\mathcal{K}(X^I)} \equiv i(\bar{X}^I F_I - X^I \bar{F}_I). \quad (4.10)$$

For the prepotential F given in (4.2) this takes the form:

$$e^{-\mathcal{K}(X^I)} = \frac{2p^1}{\phi^0}(m + E_2(\tau) + E_2(-\bar{\tau})), \quad (4.11)$$

so that

$$M(\tau, \bar{\tau}) = \frac{1}{2^{13}} \frac{\phi^0}{p^1} e^{-\mathcal{K}(\phi^I)}. \quad (4.12)$$

We make a small digression to note that, although we work with the specific details of the $\mathcal{N} = 4$ theory, most of the methods and arguments of this section extend to more general $\mathcal{N} = 2$ theories. Indeed the physical origin of τ as the off-shell fluctuations of the axion-dilaton (discussed in section 3) motivates the following form for the measure:

$$M(\tau, \bar{\tau}) = \sum_{r, \bar{r}=0}^{\infty} \widetilde{M}(r, \bar{r}) q^r \bar{q}^{\bar{r}}, \quad (4.13)$$

which is indeed satisfied for the $K3 \times T^2$ example, and is consistent with the fact that M approaches a constant as $\tau_2 \rightarrow \infty$, consistent with the large-charge limit.

The function $\mathcal{F}^{(1)}$ has a Fourier expansion in powers of $q = e^{2\pi i \tau}$:

$$e^{-\mathcal{F}^{(1)}(\tau)} = \sum_{p=-1}^{\infty} d(p) q^p, \quad (4.14)$$

with $d(p)$ for positive p being the number of instantons with charge p . Combining the measure factor (4.8), (4.9), we have (with $N_0 = 2^{-12}$):

$$\begin{aligned} M(\tau, \bar{\tau}) e^{-\mathcal{F}^{(1)}(\tau) - \mathcal{F}^{(1)}(-\bar{\tau})} &= N_0 \sum_{p, \bar{p}=-1}^{\infty} (m - p - \bar{p}) d(p) d(\bar{p}) q^p \bar{q}^{\bar{p}}, \\ &= N_0 \sum_{p, \bar{p}=-1}^{\infty} (m - p - \bar{p}) d(p) d(\bar{p}) e^{2\pi i(p-\bar{p})\tau_1} e^{-2\pi(p+\bar{p})\tau_2}. \end{aligned} \quad (4.15)$$

We now plug in the expansion (4.15) in the quantum entropy integral (4.1). For each term in this series, we can complete the square in τ_1 to get a quadratic Gaussian integrand. If we perform the τ_1 integral naively over the real line, each term in the above series would lead to an integral over τ_2 of the form (2.10). It would seem that we get an infinite series of I -Bessel functions for $\widehat{W}(n, \ell, m)$. We remind the reader that it is not surprising to find an infinite series of Bessel functions — indeed the discussion of section 2 shows

that the microscopic degeneracy has the same structure with the Bessel functions having successively sub-leading arguments. We find, however, that the arguments of the Bessel functions here decrease (as we expect) up to a point, but then increase indefinitely, thus showing that this sum is not convergent!

A solution to this puzzle was presented recently in [36] by making a choice of contour γ in (4.1) and analyzing the contributions to the degeneracies from each term in the Fourier expansion. With this choice of contour, almost all of the infinite number of Bessel functions turn out to be highly suppressed, and one is left with a finite number of I -Bessel functions, consistent with the structure of the leading $c = 1$ term of the Rademacher expansion (2.9). We now review this analysis, and use the contour prescription of [36] to make a detailed comparison between the expansion of the integral (4.7) and the $c = 1$ term of the Rademacher expansion (2.9). In making this comparison, we provide some clarifications about the details of the analysis of [36], which we present in appendix B. We find, at the end of our analysis, that the two expansions actually agree in great detail, in the appropriate regime of validity, including the integer coefficients of the Bessel functions! At first sight this observation may seem to be a pleasant surprise about this particular $\mathcal{N} = 4$ string theory, but as we sketch in the introduction, it can be understood as a reflection of the deeper and broader ideas of [9, 15], namely that worldsheet instanton degeneracies encode the microscopic black hole degeneracies in a very precise manner.

The analysis begins with the expansion (4.15) in the expression (4.1). Splitting the contour γ into two contours γ_1, γ_2 for the τ_1 and τ_2 integrals, respectively, and completing the square in each term, we obtain:

$$\begin{aligned} \widehat{W}(n, \ell, m) = N_0 \sum_{p, \bar{p} \geq -1} (m - p - \bar{p}) d(p) d(\bar{p}) e^{i\pi(p-\bar{p})\frac{\ell}{m}} \times \\ \times \int_{\gamma_2} \frac{d\tau_2}{\tau_2^{(n_v+3)/2}} \exp \left[-\pi\tau_2 \frac{\Delta(p, \bar{p})}{m} + \frac{\pi}{\tau_2} \left(n - \frac{\ell^2}{4m} \right) \right] \times \\ \times \int_{\gamma_1} d\tau_1 \exp \left[\frac{\pi m}{\tau_2} \left(\tau_1 + i(p - \bar{p}) \frac{\tau_2}{m} - \frac{\ell}{2m} \right)^2 \right], \end{aligned} \quad (4.16)$$

where we have defined

$$\Delta(p, \bar{p}) := 4m\bar{p} - (m - (p - \bar{p}))^2. \quad (4.17)$$

(We will see in the following that the function Δ becomes precisely the polar discriminants entering the Rademacher expansion (2.9).) We now define the contours γ_1, γ_2 pertaining to the τ_1 and τ_2 integrals as [36]

$$\begin{aligned} \tau_1 = i\tau_2 u : \quad -1 + \delta \leq u \leq 1 - \delta, \\ \tau_2 : \quad \epsilon - i\infty < \tau_2 < \epsilon + i\infty, \end{aligned} \quad (4.18)$$

with δ small and positive and ϵ positive. This choice ensures that $|q| < 1$ and $|\bar{q}| < 1$ so that the Fourier expansion (4.15) is convergent. As we will see below, this choice also brings $\widehat{W}(n, \ell, m)$ to a form that is exactly of the type of the $c = 1$ term in (2.9), namely as coming from a generating function that has the elliptic transformation property of a

Jacobi form of index m . Continuing as in [36], we now define the integral

$$I_u(p, \bar{p}) = \int_{\gamma_1} d\tau_1 \exp \left[\frac{\pi m}{\tau_2} \left(\tau_1 + i(p - \bar{p}) \frac{\tau_2}{m} - \frac{\ell}{2m} \right)^2 \right]. \quad (4.19)$$

Defining $\alpha \equiv (p - \bar{p})/m$, there are two types of contributions to $\widehat{W}(n, \ell, m)$ depending on whether $|\alpha| \leq 1 - \delta$ or $|\alpha| > 1 - \delta$. The leading contributions to the sum (4.16) are for $|\alpha| \leq 1 - \delta$, and the terms for which $|\alpha| > 1 - \delta$ are exponentially suppressed. We then need to take a $\delta \rightarrow 0$ limit in the contour γ_1 . This limit is rather subtle, but it can be shown that once we take it, the leading contributions to the sum (4.16) are the ones for which $|\alpha| \leq 1$ (modulo what we call “edge-effects”, as we discuss in appendix B).

Focusing on these contributions to the quantum entropy, we may evaluate the τ_1 integral in (4.16) and we are left with the τ_2 integral. The latter will yield exponentially growing I -Bessel functions (2.10) as long as $\Delta(p, \bar{p}) < 0$. Therefore, we now have two conditions, $|\alpha| \leq 1$ and $\Delta < 0$, which can be used to bound the sums over (p, \bar{p}) . Putting these facts together leads to the following expression for \widehat{W} :

$$\begin{aligned} \widehat{W}(n, \ell, m) &\simeq N_0 \sum_{p, \bar{p} \geq -1} \sum_{\substack{-m \leq p - \bar{p} \leq m \\ \Delta(p, \bar{p}) < 0}} (m - p - \bar{p}) d(p) d(\bar{p}) e^{i\pi(p - \bar{p}) \frac{\ell}{m}} \times \\ &\times \frac{i}{\sqrt{m}} \int_{\gamma_2} \frac{d\tau_2}{\tau_2^{(n_v+2)/2}} \exp \left[-\pi\tau_2 \frac{\Delta}{m} + \frac{\pi}{\tau_2} \left(n - \frac{\ell^2}{4m} \right) \right]. \end{aligned} \quad (4.20)$$

Here the \simeq sign is taken to mean that we have thrown away exponentially suppressed contributions to the complete answer for \widehat{W} . We can now evaluate the remaining integral on the contour γ_2 , which yields a Bessel function:

$$\begin{aligned} \widehat{W}(n, \ell, m) &\simeq N_0 \sum_{p, \bar{p} \geq -1} \sum_{\substack{-m \leq p - \bar{p} \leq m \\ \Delta(p, \bar{p}) < 0}} (m - p - \bar{p}) d(p) d(\bar{p}) e^{i\pi(p - \bar{p}) \frac{\ell}{m}} \times \\ &\times \frac{2\pi}{\sqrt{m}} \left(\frac{-\Delta(p, \bar{p})/m}{n - \frac{\ell^2}{4m}} \right)^{n_v/4} I_{n_v/2} \left(2\pi \sqrt{-\frac{\Delta(p, \bar{p})}{m} \left(n - \frac{\ell^2}{4m} \right)} \right). \end{aligned} \quad (4.21)$$

The symmetry $\Delta(p, \bar{p}) = \Delta(\bar{p}, p)$ implies that one can write the above expression as a sum over $p - \bar{p}$ from 0 to m , with the replacement of the phase $e^{i\pi(p - \bar{p}) \frac{\ell}{m}}$ by $\cos(\pi(p - \bar{p}) \frac{\ell}{m})$.

To proceed further, we make the following change of variables:

$$\ell' \equiv m - (p - \bar{p}), \quad n' \equiv \bar{p}. \quad (4.22)$$

In these variables, we have $\Delta(n', \ell') = 4mn' - \ell'^2$ as anticipated, and (4.21) takes the form

$$\begin{aligned} \widehat{W}(n, \ell, m) &\simeq 2N_0 \sum_{\substack{0 \leq \ell' \leq m \\ n' \geq -1}} \sum_{4n' - \frac{\ell'^2}{m} < 0} (\ell' - 2n') d(m + n' - \ell') d(n') \cos \left(\pi(m - \ell') \frac{\ell}{m} \right) \times \\ &\times \frac{2\pi}{\sqrt{m}} \left(\frac{|4n' - \frac{\ell'^2}{m}|}{n - \frac{\ell^2}{4m}} \right)^{n_v/4} I_{n_v/2} \left(2\pi \sqrt{|4n' - \frac{\ell'^2}{m}| \left(n - \frac{\ell^2}{4m} \right)} \right). \end{aligned} \quad (4.23)$$

In this form, \widehat{W} can readily be compared to the leading Rademacher expansion for a Jacobi form of index m and weight $(3 - n_v)/2$. Indeed for such a Jacobi form, the $c = 1$ term of the Rademacher expansion (2.9), (2.12) reads

$$c(n, \ell) \simeq \frac{1}{2^{(n_v-1)/2}} \sum_{0 \leq \ell' \leq m} \sum_{4n' - \frac{\ell'^2}{m} < 0} c(n', \ell') \cos \left(\pi(m - \ell') \frac{\ell}{m} \right) \times \\ \times \frac{2\pi}{\sqrt{m}} \left(\frac{\left| 4n' - \frac{\ell'^2}{m} \right|}{n - \frac{\ell^2}{4m}} \right)^{n_v/4} I_{n_v/2} \left(2\pi \sqrt{\left| 4n' - \frac{\ell'^2}{m} \right| \left(n - \frac{\ell^2}{4m} \right)} \right). \quad (4.24)$$

We see that (4.23) has exactly the same form as (4.24) if we make the identification:

$$c(n, \ell) = (\ell - 2n) d(m + n - \ell) d(n), \quad 4mn - \ell^2 < 0, \quad n \geq -1, \quad 0 \leq \ell \leq m. \quad (4.25)$$

We read this formula as an explicit prediction for the left-hand side which are the polar coefficients $c^F(n, \ell)$ of the mock Jacobi forms (2.18) that control the single-centered black hole degeneracies. The coefficients $d(p)$ of the right hand side are the instanton degeneracies captured by the function $\mathcal{F}^{(1)}$ (4.14)

$$\frac{1}{\eta(\tau)^{24}} = \sum_{n \geq -1} d(n) q^n = q^{-1} + 24 + 324q + 3200q^2 + 25650q^3 + 176256q^4 + \dots \quad (4.26)$$

The fact that the instanton degeneracies $d(n)$ vanish for $n < -1$ is reflected in the fact that the single centered degeneracies $c^F(n, \ell)$ also vanish for $n < -1$ as we saw briefly in section 2. In the next section we see that the expansion (4.23) agrees very precisely with the Rademacher-like expansion for the Fourier coefficients $c^F(n, \ell)$ — up to an order where the latter starts to deviate from the form (4.24) due to its mock modular nature.

5 Polar terms in quarter-BPS black holes in $\mathcal{N} = 4$ theory

In this section we verify the relation (4.25) for the first few values of m . We explain that there are three sources of approximations in our derivation which impose a regime of validity for the comparison of the macroscopic and the microscopic formulas. The first source is that we have only kept the first ($c = 1$) series in the microscopic Rademacher expansion while we should really keep all the terms from $c = 1, 2, 3, \dots$. The second source, as we explained in section 2, is that the effects of the shadow of the mock modular forms (although small to leading order) can become relevant at a certain sub-leading order. The third, as we explain in detail in appendix B, is what we call “edge-effects” in the evaluation of the two-dimensional integral which is the result of the localized supergravity path integral. The first source can be controlled in a fairly straightforward manner but typically this is the smallest effect. The second source is an interesting problem in analytic number theory, and the third is a problem for us to better define our contour prescription in supergravity. We leave these two problems for the future. We now explain these three effects with examples.

We begin with $m = 1$. We have:

$$\psi_1^F(\tau, z) = \frac{1}{\eta(\tau)^{24}} (3E_4(\tau)A(\tau, z) - 648\mathcal{H}_1(\tau, z)), \quad (5.1)$$

whose Fourier expansion begins as:

$$\begin{aligned}\psi_1^F(\tau, z) = & (3\zeta + 48 + 3\zeta^{-1})q^{-1} + (48\zeta^2 + 600\zeta - 648 + 600\zeta^{-1} + 48\zeta^{-2}) + \\ & (3\zeta^3 - 648\zeta^2 + 25353\zeta - 50064 + 25353\zeta^{-1} - 648\zeta^{-2} + 3\zeta^{-3})q + \\ & (600\zeta^3 - 50064\zeta^2 + 561576\zeta - 1127472 + 561576\zeta^{-1} - 50064\zeta^{-2} + 600\zeta^{-3})q^2 + \dots\end{aligned}\quad (5.2)$$

The polar terms are $(n, \ell) = (-1, 1)$, $(-1, 0)$, and $(0, 1)$ or equivalently $(\Delta, \ell) = (-5, 1)$, $(-4, 0)$, $(-1, 1)$. The corresponding coefficients $c_1^F(n, \ell)$ are⁴ [55]:

$$c_1^F(-1, 1) = 3, \quad c_1^F(-1, 0) = 48, \quad c_1^F(0, 1) = 600. \quad (5.3)$$

The corresponding combinations of the $(\ell - 2n) d(m + n - \ell) d(n)$ are:

$$(n, \ell) = (-1, 1) : 3, \quad (n, \ell) = (-1, 0) : 48, \quad (n, \ell) = (0, 1) : 576. \quad (5.4)$$

We see that the first two coefficients agree, and the third does not. This is exactly what we expect, as we explained at the end of section 2. Indeed we have made an approximation in the Rademacher expansion keeping only the leading $c = 1$ term, and we have⁵

$$\frac{\widehat{W}(n, 0, 1)}{4\pi N_0} = 3\left(\frac{5}{n}\right)^{23/4} I_{23/2}(2\pi\sqrt{5n}) + 48\left(\frac{4}{n}\right)^{23/4} I_{23/2}(2\pi\sqrt{4n}) + 576\left(\frac{1}{n}\right)^{23/4} I_{23/2}(2\pi\sqrt{n}), \quad (5.5)$$

while the $c = 1$ term of the Rademacher expansion of a Jacobi form with the polar coefficients (5.3) is:

$$\frac{c_1^F(n, 0)}{4\pi N_0} = 3\left(\frac{5}{n}\right)^{23/4} I_{23/2}(2\pi\sqrt{5n}) + 48\left(\frac{4}{n}\right)^{23/4} I_{23/2}(2\pi\sqrt{4n}) + 600\left(\frac{1}{n}\right)^{23/4} I_{23/2}(2\pi\sqrt{n}), \quad (5.6)$$

with $N_0 = 2^{-12}$. The $c = 2$ series in the expansion (2.9) starts with $I_{23/2}(2\pi\sqrt{5n/4})$ which is larger than the last term in (5.6), and therefore we do not expect an agreement at this order for the last coefficients in (5.5) and (5.6). This is one of the issues that we need to be careful about in our comparison.

Secondly, as we stressed at the end of section 2, we also need to be careful about the interference of the mock nature of the functions ψ_m^F . The first time⁶ we see this interference

⁴We note that there is a textual error in the appendix of [55]. In the first paragraph of the appendix, it says that the coefficients $c_m^F(n, \ell)$ of the mock Jacobi forms are presented for the first four values of m , while what is really presented is $d(n, \ell, m) = (-1)^\ell c_m^F(n, \ell)$ to emphasize the positivity of those numbers. In particular, the polar coefficients $c_m^F(n, \ell)$ (i.e. with $4mn - \ell^2 < 0$) are strictly positive.

⁵We do the comparison at $\ell = 0$ for simplicity.

⁶We find experimentally that for $m = 1, 2$ the two expansions agree even including the mock piece, but we believe this is an accident, which will be explained if we work out the asymptotic expansion of the corresponding mock Jacobi form in detail. This fact is highlighted by the use of a * in the tables below.

is for $m = 3$, where we have:⁷

$$\begin{aligned}
 \frac{\sqrt{3}}{4\pi N_0} \widehat{W}(n, 0, 3) = & 5 \left(\frac{7}{n}\right)^{23/4} I_{23/2} \left(2\pi\sqrt{7n}\right) + 96 \left(\frac{16}{3n}\right)^{23/4} I_{23/2} \left(2\pi\sqrt{\frac{16}{3}n}\right) \\
 & + 972 \left(\frac{13}{3n}\right)^{23/4} I_{23/2} \left(2\pi\sqrt{\frac{13}{3}n}\right) + 6400 \left(\frac{4}{n}\right)^{23/4} I_{23/2} \left(2\pi\sqrt{4n}\right) \\
 & + 1728 \left(\frac{3}{n}\right)^{23/4} I_{23/2} \left(2\pi\sqrt{3n}\right) + 15552 \left(\frac{4}{3n}\right)^{23/4} I_{23/2} \left(2\pi\sqrt{\frac{4}{3}n}\right) \\
 & + 76800 \left(\frac{1}{3n}\right)^{23/4} I_{23/2} \left(2\pi\sqrt{\frac{1}{3}n}\right). \tag{5.7}
 \end{aligned}$$

Correspondingly, the $c = 1$ term of the Rademacher expansion (2.9) for $m = 3$ is:

$$\begin{aligned}
 \frac{\sqrt{3}}{4\pi N_0} c_3^F(n, 0) = & 5 \left(\frac{7}{n}\right)^{23/4} I_{23/2} \left(2\pi\sqrt{7n}\right) + 96 \left(\frac{16}{3n}\right)^{23/4} I_{23/2} \left(2\pi\sqrt{\frac{16}{3}n}\right) \\
 & + 972 \left(\frac{13}{3n}\right)^{23/4} I_{23/2} \left(2\pi\sqrt{\frac{13}{3}n}\right) + 6404 \left(\frac{4}{n}\right)^{23/4} I_{23/2} \left(2\pi\sqrt{4n}\right) \\
 & + 1728 \left(\frac{3}{n}\right)^{23/4} I_{23/2} \left(2\pi\sqrt{3n}\right) + 15600 \left(\frac{4}{3n}\right)^{23/4} I_{23/2} \left(2\pi\sqrt{\frac{4}{3}n}\right) \\
 & + 85176 \left(\frac{1}{3n}\right)^{23/4} I_{23/2} \left(2\pi\sqrt{\frac{1}{3}n}\right). \tag{5.8}
 \end{aligned}$$

The $c = 2$ term of the Rademacher expansion starts with $I_{23/2}(2\pi\sqrt{7n/4})$, and we should ignore terms of that order, i.e. the last two Bessels in (5.8). However, we still see a disagreement for the Bessel $I_{23/2}(2\pi\sqrt{4n})$. This is precisely the interference from the mixed mock Jacobi form $\varphi_{2,3}^{\text{opt}}(\tau, z)/\eta(\tau)$ ²⁴. Therefore we should only expect agreement up to the Bessel functions $I_{23/2}(2\pi\sqrt{4n})$. In the expressions (5.7), (5.8), this means that we should not expect a matching of the coefficients for the fourth terms, 6400 vs. 6404.

Thirdly, in deriving our Rademacher-like expression from the supergravity path integral, we made a choice of contour in (4.18). As explained in appendix B, there are “edge-effects” in this contour that we have not taken into account properly here. These may go towards explaining the boxed discrepancies in the tables below for $m = 5$ and $m = 7$. We believe a more detailed analysis of the integral $I_u(p, \bar{p})$ in (4.19) would resolve these discrepancies.

We checked up to $m = 7$ that this kind of an agreement holds exactly after taking into account these three effects. We present the data in the form of tables below.

Legend for tables. The pair (n, ℓ) satisfies the conditions in (4.25), i.e. $n \geq -1$, $0 \leq \ell \leq m$ and $(4mn - \ell^2) = \Delta < 0$. The third column is the coefficient $c^F(n, \ell)$ of the mock Jacobi forms ψ_m^F (2.18). Recall that the black hole exists for positive values of Δ and

⁷Again, we do the comparison at $\ell = 0$ for simplicity.

the degeneracy $c_m(n, \ell)$ is controlled by the polar coefficients through an expansion of the type (4.24). (Essentially a polar term labelled by Δ enters the analytic formula for the degeneracy $c_m(n, \ell)$ for $4mn - \ell^2 > 0$ at an order $\exp(2\pi|\Delta|(4n - \ell^2))$.) The coefficients below the horizontal line have deviations from their true values because we have only included the $c = 1$ series of the Rademacher expansion, while at these orders we should necessarily start including the $c \geq 2$ series. We indicate in bold face when the Rademacher expansion cannot be trusted because we have treated a mock Jacobi form as a true Jacobi form. (For $m = 1, 2$ the coefficient still agree — which we indicate by a *.) As we see clearly in the tables, the deviations for the bold-faced coefficients are small and should be resolved by including the effects of the shadow. The boxed values indicate possible edge-effects in the contour prescription.

m = 1:

Δ	(n, ℓ)	$c_1(n, \ell)$	$(\ell - 2n) d(1 + n - \ell) d(n)$
−5	(−1, 1)	3	3
−4	(−1, 0)	48*	48
−1	(0, 1)	600	576

m = 2:

Δ	(n, ℓ)	$c_2(n, \ell)$	$(\ell - 2n) d(2 + n - \ell) d(n)$
−12	(−1, 2)	4	4
−9	(−1, 1)	72	72
−8	(−1, 0)	648*	648
−4	(0, 2)	1152	1152
−1	(0, 1)	8376	7776

m = 3:

Δ	(n, ℓ)	$c_3(n, \ell)$	$(\ell - 2n) d(3 + n - \ell) d(n)$
−21	(−1, 3)	5	5
−16	(−1, 2)	96	96
−13	(−1, 1)	972	972
−12	(−1, 0)	6404	6400
−9	(0, 3)	1728	1728
−4	(0, 2)	15600	15552
−1	(0, 1)	85176	76800

m = 4:

Δ	(n, ℓ)	$c_4(n, \ell)$	$(\ell - 2n) d(4 + n - \ell) d(n)$
-32	(-1, 4)	6	6
-25	(-1, 3)	120	120
-20	(-1, 2)	1296	1296
-17	(-1, 1)	9600	9600
-16	(0, 4)	2304	2304
-16	(-1, 0)	51396	51300
-9	(0, 3)	23328	23328
-4	(0, 2)	154752	153600
-1	(0, 1)	700776	615600

m = 5:

Δ	(n, ℓ)	$c_5(n, \ell)$	$(\ell - 2n) d(5 + n - \ell) d(n)$
-45	(-1, 5)	7	7
-36	(-1, 4)	144	144
-29	(-1, 3)	1620	1620
-25	(0, 5)	2880	2880
-24	(-1, 2)	12800	12800
-21	(-1, 1)	76955	76950
-20	(-1, 0)	353808	352512
-16	(0, 4)	31104	31104
-9	(0, 3)	230472	230400
-5	(1, 5)	315255	314928
-4	(0, 2)	1246800	1231200
-1	(0, 1)	4930920	4230144

m = 6:

Δ	(n, ℓ)	$c_6(n, \ell)$	$(\ell - 2n) d(6 + n - \ell) d(n)$
-60	(-1, 6)	8	8
-49	(-1, 5)	168	168
-40	(-1, 4)	1944	1944
-36	(0, 6)	3456	3456
-33	(-1, 3)	16000	16000
-28	(-1, 2)	102600	102600
-25	(0, 5)	38880	38880
-25	(-1, 1)	528888	528768
-24	(-1, 0)	2160240	2147440
-16	(0, 4)	307200	307200
-12	(1, 6)	419904	419904
-9	(0, 3)	1848528	1846800
-4	(0, 2)	8615040	8460288
-1	(0, 1)	30700200	25769280
-1	(1, 5)	3118776	3110400

$\mathbf{m} = 7$:

Δ	(n, ℓ)	$c_7(n, \ell)$	$(\ell - 2n) d(7 + n - \ell) d(n)$
-77	(-1, 7)	9	9
-64	(-1, 6)	192	192
-53	(-1, 5)	2268	2268
-49	(0, 7)	4032	4032
-44	(-1, 4)	19200	19200
-37	(-1, 3)	128250	128250
-36	(0, 6)	46656	46656
-32	(-1, 2)	705030	705024
-29	(-1, 1)	3222780	3221160
-28	(-1, 0)	11963592	11860992
-25	(0, 5)	384000	384000
-21	(1, 7)	524880	524880
-16	(0, 4)	2462496	2462400
-9	(0, 3)	12713760	12690432
-8	(1, 6)	4147848	4147200
-4	(0, 2)	52785360	51538560
-1	(0, 1)	173032104	142331904

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A Single-center degeneracies and mock Jacobi forms

In this appendix we briefly review some facts from [30] that are relevant for the discussion of section 2 and section 5. First we recall the construction of the single-centered black hole partition functions from the Igusa cusp form. The first step in [30] to analyze the single-center Fourier coefficients is to expand the microscopic partition function in $e^{2\pi i\sigma}$:

$$\frac{1}{\Phi_{10}(\tau, z, \sigma)} = \sum_{m \geq -1} \psi_m(\tau, z) e^{2\pi i m \sigma}. \quad (\text{A.1})$$

One then defines the polar part of ψ_m (with $q = e^{2\pi i\tau}$, $\zeta = e^{2\pi iz}$)

$$\psi_m^{\text{P}}(\tau, z) := \frac{p_{24}(m+1)}{\eta^{24}(\tau)} \sum_{s \in \mathbb{Z}} \frac{q^{ms^2+s} \zeta^{2ms+1}}{(1 - \zeta q^s)^2}, \quad (\text{A.2})$$

where $p_{24}(n)$ counts the number of partitions of an integer n with 24 colors. The function ψ_m^P is the average over the lattice $\mathbb{Z}\tau + \mathbb{Z}$ of the leading behavior of the function near the pole $z = 0$

$$\frac{p_{24}(m+1)}{\eta(\tau)^{24}} \frac{\zeta}{(1-\zeta)^2}. \quad (\text{A.3})$$

The function ψ_m^P is an example of an Appell-Lerch sum, and it encodes the physics of all the wall-crossings due to the decay of two-centered black holes.

The two functions ψ_m and ψ_m^P have, by construction, the same poles and residues, so the difference

$$\psi_m^F := \psi_m - \psi_m^P, \quad (\text{A.4})$$

called the *finite* or Fourier part of ψ_m , is holomorphic in z , and has an unambiguous Fourier expansion:

$$\psi_m^F(\tau, z) = \sum_{n, \ell} c_m^F(n, \ell) q^n \zeta^\ell. \quad (\text{A.5})$$

The indexed degeneracies of the single-centered black hole of magnetic charge invariant $Q_m^2/2 = m$, as defined by the attractor mechanism, are related to the Fourier coefficients of the function ψ_m^F as $d(n, \ell, m) = (-1)^{\ell+1} c_m^F(n, \ell)$, the overall sign coming from an analysis of the fermion zero modes described in [38]. The statement of the main theorem of ([30], Chapter 8) is that the single-center black hole partition function $\psi_m^F(\tau, z)$ is a *mock Jacobi form*.

What this means is that ψ^F has the same elliptic transformation property (2.5) as a regular Jacobi form governed by the index m . Its modular transformation property (2.4), however, is modified. The lack of modularity is governed by the explicit function called the *shadow*:

$$\psi^S(\tau, z) = \frac{1}{\eta(\tau)^{24}} \sum_{\ell \in \mathbb{Z}/2m\mathbb{Z}} \vartheta_{m, \ell}^*(\tau, 0) \vartheta_{m, \ell}(\tau, z), \quad (\text{A.6})$$

where the operation $*$ is such that, a modular form g of weight w obeys

$$(4\pi\tau_2)^w \frac{\partial g^*(\tau)}{\partial \bar{\tau}} = -2\pi i \overline{g(\tau)}. \quad (\text{A.7})$$

The function

$$\widehat{\psi^F}(\tau, z) = \psi^F(\tau, z) + \psi^S(\tau, z), \quad (\text{A.8})$$

called the *completion* of ψ^F , transforms as a Jacobi form of weight -10 and index m , but it is not holomorphic. It obeys the holomorphic anomaly equation:

$$(4\pi\tau_2)^{1/2} \frac{\partial \widehat{\psi^F}(\tau, z)}{\partial \bar{\tau}} = -2\pi i \frac{1}{\eta(\tau)^{24}} \sum_{\ell \in \mathbb{Z}/2m\mathbb{Z}} \overline{\vartheta_{m, \ell}(\tau, 0)} \vartheta_{m, \ell}(\tau, z). \quad (\text{A.9})$$

Now we briefly present some relevant facts about the growth of the coefficients of the mock Jacobi forms ψ_m^F . By multiplying ψ_m^F by the function $\eta(\tau)^{24}$, we get a function $\varphi_{2, m}^{\text{mock}} = \eta^{24} \psi_m^F$ which is a mock Jacobi form of weight 2 and index m . It was shown

in ([30], Chapters 9, 10) that $\varphi_{2,m}$ can be written⁸ as a linear combination of a (true) weak Jacobi form and a mock Jacobi form

$$\varphi_{2,m}^{\text{mock}}(\tau, z) = \varphi_{2,m}^{\text{true}}(\tau, z) + \varphi_{2,m}^{\text{opt}}(\tau, z), \quad (\text{A.10})$$

such that the mock Jacobi form $\varphi_{2,m}^{\text{opt}}$ has *optimal growth*. This means that the Fourier-Jacobi coefficients of $\varphi_{2,m}^{\text{opt}}(\tau, z)$ grow at most as

$$c^{\text{opt}}(n, \ell) \sim \exp\left(\frac{\pi}{m} \sqrt{4mn - \ell^2}\right). \quad (\text{A.11})$$

If we look at the Rademacher expansion (2.9), the growth (A.11) is the smallest possible one, governed by the value of $|\tilde{\Delta}| = 1$. In fact, for m a prime power, the coefficients of the optimal mock Jacobi form has only polynomial growth.

B The I_u integral

In this appendix, we perform a detailed analysis of the I_u integral defined in (4.19). On the contour γ_1 defined in (4.18), we have:

$$\begin{aligned} I_u(p, \bar{p}) &= i\tau_2 \int_{-1+\delta}^{1-\delta} du \exp\left[-\pi m \tau_2 \left(u + \alpha - \frac{\ell}{2im\tau_2}\right)^2\right], \\ &= \frac{1}{2} \sqrt{\frac{\tau_2}{m}} \left[\text{Erfi}\left(\frac{\sqrt{\pi}(\ell - 2i\tau_2 m(\alpha - 1 + \delta))}{2\sqrt{\tau_2 m}}\right) - \text{Erfi}\left(\frac{\sqrt{\pi}(\ell - 2i\tau_2 m(\alpha + 1 - \delta))}{2\sqrt{\tau_2 m}}\right) \right], \end{aligned} \quad (\text{B.1})$$

where $\text{Erfi}(x)$ is the imaginary error function $\text{Erfi}(x) = \text{Erf}(ix)/i$. Above we have defined $\alpha \equiv (p - \bar{p})/m$, and δ is a small, positive constant parametrizing the contour γ_1 .

We now take $\text{Re}(\tau_2) = \epsilon$ to be very large and use the Taylor series of the imaginary error function in this regime. Depending on the value of $|\alpha|$, we obtain three results: first for $|\alpha| < 1 - \delta$, where the Taylor expansion gives:

$$\begin{aligned} I_u^{|\alpha| < 1 - \delta} &= i\sqrt{\frac{\tau_2}{m}} + \exp\left[\pi \frac{(\ell - 2i\tau_2 m(\alpha - 1 + \delta))^2}{4\tau_2 m}\right] \left(\frac{i}{2\pi m(\alpha - 1 + \delta)} + \mathcal{O}\left(\frac{1}{\epsilon}\right)\right) \\ &\quad - \exp\left[\pi \frac{(\ell - 2i\tau_2 m(\alpha + 1 - \delta))^2}{4\tau_2 m}\right] \left(\frac{i}{2\pi m(\alpha + 1 - \delta)} + \mathcal{O}\left(\frac{1}{\epsilon}\right)\right). \end{aligned} \quad (\text{B.2})$$

Secondly, for $|\alpha| = 1 - \delta$:

$$\begin{aligned} I_u^{\alpha = \pm(1-\delta)} &= \frac{i}{2} \sqrt{\frac{\tau_2}{m}} \pm \frac{\ell}{2m} \\ &\quad - \exp\left[\pi \frac{(\ell \pm 4i\tau_2 m(\delta - 1))^2}{4\tau_2 m}\right] \left(\frac{i}{4\pi m(1 - \delta)} + \mathcal{O}\left(\frac{1}{\epsilon}\right)\right). \end{aligned} \quad (\text{B.3})$$

Lastly, for $|\alpha| > 1 - \delta$:

$$\begin{aligned} I_u^{|\alpha| > 1 - \delta} &= \exp\left[\pi \frac{(\ell - 2i\tau_2 m(\alpha - 1 + \delta))^2}{4\tau_2 m}\right] \left(\frac{i}{2\pi m(\alpha - 1 + \delta)} + \mathcal{O}\left(\frac{1}{\epsilon}\right)\right) \\ &\quad - \exp\left[\pi \frac{(\ell - 2i\tau_2 m(\alpha + 1 - \delta))^2}{4\tau_2 m}\right] \left(\frac{i}{2\pi m(\alpha + 1 - \delta)} + \mathcal{O}\left(\frac{1}{\epsilon}\right)\right). \end{aligned} \quad (\text{B.4})$$

⁸Recall that the definition of a mock Jacobi form only holds modulo the addition of a true Jacobi form.

Focusing on the case where $|\alpha| < 1 - \delta$, we can use the above expressions for I_u in (4.16). The τ_2 integral is now on a contour where $\epsilon \gg 1$, but since the only pole in the τ_2 complex plane sits at the origin, we can safely deform it back to ϵ small and still positive. On this contour, we find using (2.10):

$$\begin{aligned}
 \widehat{W}^{|\alpha| < 1 - \delta} = & N_0 \sum_{p, \bar{p} \geq -1} \left[\sum_{\substack{-m(1-\delta) < p - \bar{p} < m(1-\delta) \\ \Delta(p, \bar{p}) < 0}} (m - p - \bar{p}) d(p) d(\bar{p}) e^{i\pi(p - \bar{p}) \frac{\ell}{m}} \times \right. \\
 & \times \frac{2\pi}{\sqrt{m}} \left(\frac{|\Delta(p, \bar{p})/m|}{n - \frac{\ell^2}{4m}} \right)^{n_v/4} I_{n_v/2} \left(2\pi \sqrt{\left| \frac{\Delta(p, \bar{p})}{m} \right| \left(n - \frac{\ell^2}{4m} \right)} \right) \\
 & + \sum_{\substack{-m(1-\delta) < p - \bar{p} < m(1-\delta) \\ 4\bar{p} - 2\delta(\bar{p} - p + m) + m\delta^2 < 0}} (m - p - \bar{p}) d(p) d(\bar{p}) e^{i\pi\ell(1-\delta)} \times \\
 & \times \frac{1}{p - \bar{p} - m(1 - \delta)} \left(\frac{|4\bar{p} - 2\delta(\bar{p} - p + m) + m\delta^2|}{n} \right)^{(n_v+1)/4} \times \\
 & \times I_{(n_v+1)/2} \left(2\pi \sqrt{|4\bar{p} - 2\delta(\bar{p} - p + m) + m\delta^2| n} \right) \\
 & + \sum_{\substack{-m(1-\delta) < p - \bar{p} < m(1-\delta) \\ 4p - 2\delta(p - \bar{p} + m) + m\delta^2 < 0}} (m - p - \bar{p}) d(p) d(\bar{p}) e^{-i\pi\ell(1-\delta)} \times \\
 & \times \frac{1}{\bar{p} - p - m(1 - \delta)} \left(\frac{|4p - 2\delta(p - \bar{p} + m) + m\delta^2|}{n} \right)^{(n_v+1)/4} \times \\
 & \left. \times I_{(n_v+1)/2} \left(2\pi \sqrt{|4p - 2\delta(p - \bar{p} + m) + m\delta^2| n} \right) \right]. \tag{B.5}
 \end{aligned}$$

The case $|\alpha| = 1 - \delta$ contributes to \widehat{W} as:

$$\begin{aligned}
 \widehat{W}^{\alpha = \pm(1-\delta)} = & N_0 \sum_{p \geq -1} \left[\sum_{4mp - m^2\delta^2 < 0} (m\delta - 2p) d(p) d(p + m - m\delta) e^{i\pi(1-\delta)\ell} \times \right. \\
 & \times \frac{\pi}{\sqrt{m}} \left(\frac{|4p - m\delta^2|}{n - \frac{\ell^2}{4m}} \right)^{n_v/4} I_{n_v/2} \left(2\pi \sqrt{|4p - m\delta^2| \left(n - \frac{\ell^2}{4m} \right)} \right) \\
 & \pm \frac{\ell}{2m} \sum_{4mp - m^2\delta^2 < 0} (m\delta - 2p) d(p) d(p + m - m\delta) e^{i\pi(1-\delta)\ell} \times \\
 & \times 2\pi \left(\frac{|4p - m\delta^2|}{n} \right)^{(n_v+1)/4} I_{(n_v+1)/2} \left(2\pi \sqrt{|4p - m\delta^2| n} \right) \\
 & + \sum_{4p + 4m - 8m\delta + 3m\delta^2 < 0} (m\delta - 2p) d(p) d(p + m - m\delta) e^{\mp i\pi(1-\delta)\ell} \times \\
 & \times \frac{1}{2m(\delta - 1)} \left(\frac{|4p + 4m - 8m\delta + 3m\delta^2|}{n} \right)^{(n_v+1)/4} \times \\
 & \left. \times I_{(n_v+1)/2} \left(2\pi \sqrt{|4p + 4m - 8m\delta + 3m\delta^2| n} \right) \right]. \tag{B.6}
 \end{aligned}$$

At this point, we can put the three contributions together to obtain \widehat{W} for $|\alpha| \leq 1 - \delta$:

$$\begin{aligned}
 \widehat{W}^{|\alpha| \leq 1-\delta} = & N_0 \sum_{p, \bar{p} \geq -1 - m(1-\delta)} \sum_{\substack{\leq p - \bar{p} \leq m(1-\delta) \\ \Delta(p, \bar{p}) < 0}} (m - p - \bar{p}) d(p) d(\bar{p}) e^{i\pi(p-\bar{p})\frac{\ell}{m}} \times \\
 & \times \frac{2\pi}{\sqrt{m}} \left(\frac{|\Delta(p, \bar{p})/m|}{n - \frac{\ell^2}{4m}} \right)^{n_v/4} I_{n_v/2} \left(2\pi \sqrt{\left| \frac{\Delta(p, \bar{p})}{m} \right| \left(n - \frac{\ell^2}{4m} \right)} \right) \\
 & + N_0 \sum_{p, \bar{p} \geq -1 - m(1-\delta)} \sum_{\substack{< p - \bar{p} < m(1-\delta) \\ 4\bar{p} - 2\delta(\bar{p} - p + m) + m\delta^2 < 0}} (m - p - \bar{p}) d(p) d(\bar{p}) e^{i\pi\ell(1-\delta)} \times \\
 & \times \frac{1}{p - \bar{p} - m(1-\delta)} \left(\frac{|4\bar{p} - 2\delta(\bar{p} - p + m) + m\delta^2|}{n} \right)^{(n_v+1)/4} \times \\
 & \times I_{(n_v+1)/2} \left(2\pi \sqrt{|4\bar{p} - 2\delta(\bar{p} - p + m) + m\delta^2| n} \right) \\
 & + N_0 \sum_{p, \bar{p} \geq -1 - m(1-\delta)} \sum_{\substack{< p - \bar{p} < m(1-\delta) \\ 4p - 2\delta(p - \bar{p} + m) + m\delta^2 < 0}} (m - p - \bar{p}) d(p) d(\bar{p}) e^{-i\pi\ell(1-\delta)} \times \\
 & \times \frac{1}{\bar{p} - p - m(1-\delta)} \left(\frac{|4p - 2\delta(p - \bar{p} + m) + m\delta^2|}{n} \right)^{(n_v+1)/4} \times \\
 & \times I_{(n_v+1)/2} \left(2\pi \sqrt{|4p - 2\delta(p - \bar{p} + m) + m\delta^2| n} \right) \\
 & + N_0 \sum_{p \geq -1} \sum_{4p+4m-8m\delta+3m\delta^2 < 0} (m\delta - 2p) d(p) d(p+m-m\delta) \times \\
 & \times \frac{\cos(\pi(\delta-1)\ell)}{m(\delta-1)} \left(\frac{|4p+4m-8m\delta+3m\delta^2|}{n} \right)^{(n_v+1)/4} \times \\
 & \times I_{(n_v+1)/2} \left(2\pi \sqrt{|4p+4m-8m\delta+3m\delta^2| n} \right). \tag{B.7}
 \end{aligned}$$

The remaining contributions to \widehat{W} are the ones for which $|\alpha| > 1 - \delta$. Comparing (B.2) and (B.4), it is clear that they will give the same type of contributions as the second and third sums of Bessel functions above, albeit with a different range for p, \bar{p} . Putting everything together, we find the following expression:

$$\begin{aligned}
 \widehat{W}(n, \ell, m) = & N_0 \sum_{p, \bar{p} \geq -1 - m(1-\delta)} \sum_{\substack{\leq p - \bar{p} \leq m(1-\delta) \\ \Delta(p, \bar{p}) < 0}} (m - p - \bar{p}) d(p) d(\bar{p}) e^{i\pi(p-\bar{p})\frac{\ell}{m}} \times \\
 & \times \frac{2\pi}{\sqrt{m}} \left(\frac{|\Delta(p, \bar{p})/m|}{n - \frac{\ell^2}{4m}} \right)^{n_v/4} I_{n_v/2} \left(2\pi \sqrt{\left| \frac{\Delta(p, \bar{p})}{m} \right| \left(n - \frac{\ell^2}{4m} \right)} \right) \\
 & + N_0 \sum_{p \geq -1} \sum_{4p+4m-8m\delta+3m\delta^2 < 0} (m\delta - 2p) d(p) d(p+m-m\delta) \times \\
 & \times \frac{\cos(\pi(\delta-1)\ell)}{m(\delta-1)} \left(\frac{|4p+4m-8m\delta+3m\delta^2|}{n} \right)^{(n_v+1)/4} \times \\
 & \times I_{(n_v+1)/2} \left(2\pi \sqrt{|4p+4m-8m\delta+3m\delta^2| n} \right)
 \end{aligned}$$

$$\begin{aligned}
& + N_0 \sum_{p, \bar{p} \geq -1} \sum_{\substack{p - \bar{p} \neq \pm m(1 - \delta) \\ 4\bar{p} - 2\delta(\bar{p} - p + m) + m\delta^2 < 0}} (m - p - \bar{p}) d(p) d(\bar{p}) \times \\
& \times \frac{2 \cos(\pi(\delta - 1)\ell)}{p - \bar{p} - m(1 - \delta)} \left(\frac{|4\bar{p} - 2\delta(\bar{p} - p + m) + m\delta^2|}{n} \right)^{(n_v+1)/4} \times \\
& \times I_{(n_v+1)/2} \left(2\pi \sqrt{|4\bar{p} - 2\delta(\bar{p} - p + m) + m\delta^2|n} \right). \tag{B.8}
\end{aligned}$$

In section 4 (and in [36]), it was argued that, in the limit where $\delta \rightarrow 0$, only the first sum of Bessel functions (the ones with weight $n_v/2$) contribute to $\widehat{W}(n, \ell, m)$ and that the second and third contributions to (B.8) are all exponentially suppressed. Indeed, the first term is unambiguous when taking the limit $\delta \rightarrow 0$, and yields precisely the expression (4.21). However, this limit is less trivial for the Bessel functions of integer weight $(n_v + 1)/2$. In fact, the contribution from these Bessels depends *sensitively* on how δ goes to zero, and on the behavior of the product $m\delta$ in this limit. Since δ is a parameter introduced for the contour γ_1 , this means that a particular choice of γ_1 can pick up additional contributions from the integer-weight Bessel functions, which may then become comparable to the Bessels of half-integer weight. This is what we call “edge-effects”, arising from the limit $\delta \rightarrow 0$ in the contour γ_1 (4.18).

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